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On models of certain p,q-algebra representations: The quantum Euclidean algebra $\mathcal{E}_{p,q}(2)$

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Abstract

We construct one and two variable models of the p,q-algebra $\mathcal{E}_{p,q}(2)$ in terms of ladder operators. We define two p,q-analogues of the exponential function mapping and using these show that matrix elements of these operators can be represented in terms of p,q-series. Further, we exhibit that these matrix elements satisfy certain biorthogonality relations.

Keywords: Quantum algebra $\mathcal{E}_{p,q}(2)$; p,q-Hypergeometric series

1. Introduction

We begin a study of models of certain p,q-algebra representations. In this paper we shall consider the p,q-algebra $\mathcal{E}_{p,q}(2)$, the two parameter quantum Euclidean algebra. The approach to p,q-algebras is to regard them as p,q-analogues of Lie algebras. We introduce two p,q-analogues of the exponential function mapping, denoted by $e_{p,q}$ and $E_{p,q}$. By replacing the usual exponential mapping from a Lie algebra to the Lie group by these p,q-exponential mappings, we derive eight types of matrix elements on the group arising from an irreducible representation. These matrix elements are expressible in terms of generalized p,q-series. We further show that each of these eight types of matrix elements gives rise to a two variable model of irreducible representations of $\mathcal{E}_{p,q}(2)$.

The present work is motivated by and is an extension of the corresponding work for models of q-algebra representations; see Kalnins, Miller and Mukherjee [7–9], and Floreanini and Vinet [3]. Work on two parameter quantum algebras and its relation with the p,q-series also exists in literature, see for example, Burban and Klimyk [1], Chakrabarti and Jagannathan [2], Floreanini, Lapointe and Vinet [4], Jagannathan and Rao [6] and Sahai and Yadav [11], to name a few. Section wise treatment is as follows.

In Section 2, we collect some basic definitions of p,q-special function theory. We define two p,q-analogues $e_{p,q}$ and $E_{p,q}$ of the exponential operator. In Section 3, we introduce the p,q-algebra $\mathcal{E}_{p,q}(2)$ and give a one-variable

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realization of its generators in terms of p,q-dilation operators. Using the p,q-exponential operators, we are led to eight types of matrix elements on the representation space, which are expressible in terms of generalized p,q-series.

In Section 4, we find several operator identities with the help of one-variable function space model and utilize them to form a two-variable model through each family of matrix elements. Finally, in Section 5, we show that the matrix elements computed earlier satisfy certain biorthogonality relations.

2. Preliminaries

The p,q-hypergeometric series ${}_n\psi_{n-1}$ is defined by [1]

$$ {}_n\psi_{n-1}(a_1,\dots,a_n;b_1,\dots,b_{n-1};p,q;z)=\sum_{m=0}^{\infty}\frac{([a_1]_{p,q})_m\cdots([a_n]_{p,q})_m}{([b_1]_{p,q})_m\cdots([b_{n-1}]_{p,q})_m}\frac{z^m}{[m]_{p,q}!}, \tag{1}$$

where $p>0$, $q>0$ and $pq<1$, and $([a]_{p,q})_m$ is the p,q-shifted factorial defined by

$$ \big([a]_{p,q}\big)_m=\begin{cases}[a]_{p,q}[a+1]_{p,q}\cdots[a+m-1]_{p,q}, & \text{if } m=1,2,\dots,\\ 1, & \text{if } m=0,\end{cases} \tag{2}$$

where $[a]_{p,q}$ is the p,q-analogue of $a\in\mathbb{C}$, given by

$$ [a]_{p,q}=\frac{p^{a/2}-q^{-a/2}}{p^{1/2}-q^{-1/2}} \tag{3}$$

and for $m\in\mathbb{Z}$

$$ [m]_{p,q}!=\big([1]_{p,q}\big)_m. \tag{4}$$

For arbitrary number of numerator and denominator parameters, the generalized p,q-hypergeometric series is given by [11],

$$ {}_r\psi_s(a_1,\dots,a_r;b_1,\dots,b_s;p,q;z)=\sum_{m=0}^{\infty}\frac{([a_1]_{p,q})_m\cdots([a_r]_{p,q})_m}{([b_1]_{p,q})_m\cdots([b_s]_{p,q})_m}\left[\frac{(-1)^m q^{-\frac12\binom{m}{2}}}{(p^{1/2}-q^{-1/2})^m}\right]^{1+s-r}\frac{z^m}{[m]_{p,q}!}, \tag{5}$$

where $p>0$, $q>0$ and $pq<1$. The series ${}_r\psi_s$ terminates if one of the numerator parameters a_i, $i=1,\dots,r$, is of the form $-m$, where $m=0,1,2,\dots$.

Further, in our work the matrix elements have been found to be in terms of special cases of rather general power series, which we denote by ${}_r\widetilde{\psi}_s$, given by

$$ \begin{aligned} &{}_r\widetilde{\psi}_s(a_1,\dots,a_r;b_1,\dots,b_s;p,q;A,z)\\ &\quad=\sum_{m=0}^{\infty}\frac{([a_1]_{p,q})_m\cdots([a_r]_{p,q})_m}{([b_1]_{p,q})_m\cdots([b_s]_{p,q})_m}\left[\frac{(-1)^m q^{-\frac12\binom{m}{2}}}{(p^{1/2}-q^{-1/2})^m}\right]^{1+s-r}A_m\frac{z^m}{[m]_{p,q}!}, \end{aligned} \tag{6}$$

where $A=\{A_m\}$ is an arbitrary sequence of complex numbers. The series (6) is a p,q-extension of a general q-series appearing in [5]. In the present paper, we shall be concerned with the particular case of (6) with $A_m=\{l^{m^2/4}\}$ which will be denoted by ${}_r\widetilde{\psi}_s\big({a_1,\dots,a_r \atop b_1,\dots,b_s};p,q;l,z\big)$.

We introduce two p,q-analogues of the exponential operator as

$$ \begin{aligned} e_{p,q}(z)&=\sum_{m=0}^{\infty}\frac{(-1)^m q^{-\frac12\binom{m}{2}}}{(p^{1/2}-q^{-1/2})^m}\frac{z^m}{[m]_{p,q}!},\quad pq<1,\ \big|q^{1/2}z\big|<1,\\ E_{p,q}(z)&=\sum_{m=0}^{\infty}\frac{(-1)^m p^{\frac12\binom{m}{2}}}{(p^{1/2}-q^{-1/2})^m}\frac{z^m}{[m]_{p,q}!},\quad pq<1,\ \text{for all } z. \end{aligned} \tag{7}$$

Although a p,q-analogue of the exponential operator exists in literature in the form [1],

$$\exp_{p,q}(z)=\sum_{m=0}^{\infty}\frac{z^m}{[m]_{p,q}!}, \tag{8}$$

we prefer to use the p,q-exponentials given by (7) on two counts: one, it satisfies

$$e_{p,q}(z)E_{p,q}(-z)=1, \tag{9}$$

and two, in the limits $p\to q^2, q\to 1$, the two q-analogues of exponential operator given by

$$e_q(z)=\sum_{m=0}^{\infty}\frac{z^m}{(q;q)_m},$$
$$E_q(z)=\sum_{m=0}^{\infty}\frac{q^{m(m-1)/2}}{(q;q)_m}z^m,$$

respectively, are recovered. Further, the p,q-analogues of the exponential operator satisfy the following relations:

$$e_{p,q}\big(p^{1/2}z\big)=e_{q,p}\big(q^{1/2}z\big),$$
$$e_{p^{-1},q^{-1}}\big(q^{1/2}z\big)=E_{p,q}\big(-p^{1/2}z\big),$$
$$E_{p,q}\big(p^{1/2}z\big)=E_{q,p}\big(q^{1/2}z\big).$$

Using the p,q-derivative operator [1],

$$D_{p,q}f(z)=\frac{T_p^{1/2}-T_q^{-1/2}}{(p^{1/2}-q^{-1/2})z}f(z), \tag{10}$$

where $T_p^{1/2}$ is the p-dilation operator defined by $T_p^{1/2}f(z)=f(p^{1/2}z)$, we have the following action of the p,q-derivative operator on the p,q-exponential operators:

$$D_{p,q}e_{p,q}\big(-z\big(p^{1/2}-q^{-1/2}\big)\big)=e_{p,q}\big(-q^{-1/2}z\big(p^{1/2}-q^{-1/2}\big)\big), \tag{11}$$
$$D_{p,q}E_{p,q}\big(-z\big(p^{1/2}-q^{-1/2}\big)\big)=E_{p,q}\big(-p^{1/2}z\big(p^{1/2}-q^{-1/2}\big)\big). \tag{12}$$

3. One-variable model of $\mathcal{E}_{p,q}(2)$ representations

It is known that the p,q-algebra $\mathcal{E}_{p,q}(2)$ is a two parameter quantum algebra [11], determined by the generators J, P_+, P_- satisfying the commutation relations

$$[J,P_+]=P_+,$$
$$[J,P_-]=-P_-,$$
$$P_+P_- - p^{-1/2}q^{1/2}P_-P_+=0. \tag{13}$$

The irreducible representations $Q(\omega,m_0)$ are characterized by $\omega,m_0\in\mathbb{C}$ with $\omega\neq 0$ and $0\leqslant \operatorname{Re} m_0<1$. The spectrum of J corresponding to $Q(\omega,m_0)$ is the set $S=\{m_0+n\colon\ n\in\mathbb{Z}\}$ and the complex representation space has the basis vectors f_m, $m\in S$, such that

$$P_+f_m=\omega f_{m+1},$$
$$P_-f_m=\left(\frac{p}{q}\right)^{m/2}\omega f_{m-1},$$
$$Jf_m=mf_m. \tag{14}$$

A canonical one-variable model of $Q(\omega,m_0)$ is given by

$$\begin{aligned} &P_+ = \omega z,\\ &P_- = \left(\frac{p}{q}\right)^{m_0/2}\frac{\omega}{z}T_p^{1/2}T_q^{-1/2},\\ &J = m_0 + z\frac{d}{dz}, \end{aligned} \tag{15}$$

acting on the space of all linear combinations of the functions z^n, $z \in \mathbb{C}$, $n \in \mathbb{Z}$, with basis vectors $f_m(z) = z^n$, where $m \in S$. The Casimir operator

$$\mathcal{C} = P_+P_-\left(\frac{q}{p}\right)^{J/2}$$

lies in the center of the algebra as it commutes with every element of the algebra. For the irreducible representation $Q(\omega, m_0)$ we have $\mathcal{C} = \omega^2 I$, where I is the identity operator.

In the classical case the matrix elements of $\mathcal{E}(2)$ in the representation $Q(\omega, m_0)$ are defined by, see [10],

$$e^{\beta P_+}e^{\alpha P_-}e^{\tau J}f_m = \sum_{m'=-\infty}^{\infty} T_{m'm}(\alpha,\beta,\tau)f_{m'},$$

where $m, m' \in S$. Using the model (15) and the p,q-analogues of the exponential operators (7), we get the following eight types of matrix elements of $Q(\omega, m_0)$. We now take $\tau = 0$, for simplicity, so that $T_{m'm}(\alpha,\beta) \simeq T_{m'm}(\alpha,\beta,0) \simeq T_{n'n}(\alpha,\beta)$, where $m = m_0 + n$, $m' = m_0 + n'$. In other words, the representations $Q(\omega, m_0)$ and $Q(\omega)$ have the same matrix elements:

$$\begin{aligned}
&(e+,e-): \quad e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e+,e-)}(\alpha,\beta)f_{n'},\\
&(e-,e+): \quad e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e-,e+)}(\alpha,\beta)f_{n'},\\
&(e+,E-): \quad e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e+,E-)}(\alpha,\beta)f_{n'},\\
&(e-,E+): \quad e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e-,E+)}(\alpha,\beta)f_{n'},\\
&(E+,e-): \quad E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E+,e-)}(\alpha,\beta)f_{n'},\\
&(E-,e+): \quad E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E-,e+)}(\alpha,\beta)f_{n'},\\
&(E+,E-): \quad E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E+,E-)}(\alpha,\beta)f_{n'},\\
&(E-,E+): \quad E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E-,E+)}(\alpha,\beta)f_{n'},
\end{aligned} \tag{16}$$

where $\alpha, \beta \in \mathbb{C}$. Using (9), we have the identities

$$\sum_{l=-\infty}^{\infty} T_{n'l}^{(e+,e-)}(\alpha,\beta)T_{ln}^{(E-,E+)}(-\beta,-\alpha) = \delta_{n'n}, \tag{17}$$

$$\sum_{l=-\infty}^{\infty} T_{n'l}^{(e-,e+)}(\alpha,\beta)T_{ln}^{(E+,E-)}(-\beta,-\alpha) = \delta_{n'n}, \tag{18}$$

$$\sum_{l=-\infty}^{\infty} T_{n'l}^{(E-,e+)}(\alpha,\beta)T_{ln}^{(E+,e-)}(-\beta,-\alpha)=\delta_{n'n}, \tag{19}$$

$$\sum_{l=-\infty}^{\infty} T_{n'l}^{(e+,E-)}(\alpha,\beta)T_{ln}^{(e-,E+)}(-\beta,-\alpha)=\delta_{n'n}. \tag{20}$$

As we shall see, these identities will be instrumental in obtaining biorthogonality relations. Using the model (15) to treat (16) as generating functions for the matrix elements, we get the following explicit expressions for the matrix elements:

$$\begin{aligned}
&T_{n'n}^{(e+,e-)}(\alpha,\beta)\\
&=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n-n'}\frac{p^{\frac14(n-n')(n+n'+1)}}{q^{\frac12 n(n-n')}[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;\frac{q}{p},\frac{\alpha\beta\omega^2p^{\frac14(2n'+1)}}{q^{\frac14(2n+1)}}\right)\\
&=\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{q^{-\frac14(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;\frac{q}{p},\frac{\alpha\beta\omega^2p^{\frac14(2n+1)}}{q^{\frac14(2n'+1)}}\right),
\end{aligned}\tag{21}$$

$$\begin{aligned}
&T_{n'n}^{(e-,e+)}(\alpha,\beta)\\
&=\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{n-n'}\frac{p^{\frac14(n-n')(n+n'+1)}}{q^{\frac12 n(n-n')}[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;\frac{p}{q},\frac{\alpha\beta\omega^2p^{\frac14(2n+1)}}{q^{\frac14(4n-2n'+1)}}\right)\\
&=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{q^{-\frac14(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;\frac{p}{q},\frac{\alpha\beta\omega^2p^{\frac14(2n'+1)}}{q^{\frac14(4n'-2n+1)}}\right),
\end{aligned}\tag{22}$$

$$\begin{aligned}
&T_{n'n}^{(e+,E-)}(\alpha,\beta)\\
&=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n-n'}\frac{p^{\frac12 n(n-n')}}{q^{\frac14(n-n')(n+n'+1)}[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;q^2,\frac{\alpha\beta\omega^2p^{\frac12 n}}{q^{\frac12(n'+1)}}\right)\\
&=\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{q^{-\frac14(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;q^2,\frac{\alpha\beta\omega^2p^{\frac12 n}}{q^{\frac12(n'+1)}}\right),
\end{aligned}\tag{23}$$

$$\begin{aligned}
&T_{n'n}^{(e-,E+)}(\alpha,\beta)\\
&=\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{n-n'}\frac{p^{\frac14(n-n')(n+n'+1)}}{q^{\frac12 n(n-n')}[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;p^2,\frac{\alpha\beta\omega^2p^{\frac{n}{2}}}{q^{\frac12(2n-n'+1)}}\right)\\
&=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{p^{\frac14(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;p^2,\frac{\alpha\beta\omega^2p^{\frac12(2n'-n)}}{q^{\frac12(n'+1)}}\right),
\end{aligned}\tag{24}$$

$$\begin{aligned}
&T_{n'n}^{(E+,e-)}(\alpha,\beta)\\
&=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n-n'}\frac{p^{\frac14(n-n')(n+n'+1)}}{q^{\frac12 n(n-n')}[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;q^2,\frac{\alpha\beta\omega^2p^{\frac12 n'}}{q^{\frac12(n+1)}}\right)\\
&=\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{p^{\frac14(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;q^2,\frac{\alpha\beta\omega^2p^{\frac12 n'}}{q^{\frac12(n+1)}}\right),
\end{aligned}\tag{25}$$

$$\begin{aligned}
&T^{(E-,e+)}_{n'n}(\alpha,\beta)\\
&\quad=\left(\frac{-\beta\omega p^{\frac{1}{2}n}}{q^{\frac{1}{4}(n+n'+1)}(p^{1/2}-q^{-1/2})}\right)^{n-n'}\frac{1}{[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;p^2,\frac{\alpha\beta\omega^2p^{\frac{1}{2}(2n-n')}}{q^{\frac{1}{2}(n+1)}}\right)\\
&\quad=\left(\frac{-\alpha\omega}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{q^{-\frac{1}{4}(n'-n)(n'-n-1)}}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;p^2,\frac{\alpha\beta\omega^2p^{\frac{1}{2}n'}}{q^{\frac{1}{2}(2n'-n+1)}}\right),
\end{aligned}\tag{26}$$

$$\begin{aligned}
&T^{(E+,E-)}_{n'n}(\alpha,\beta)\\
&\quad=\left(\frac{-\alpha\omega p^{\frac{1}{2}n}}{q^{\frac{1}{4}(n+n'+1)}(p^{1/2}-q^{-1/2})}\right)^{n-n'}\frac{1}{[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;pq^3,\frac{\alpha\beta\omega^2p^{\frac{1}{4}(2n-1)}}{q^{\frac{1}{4}(2n'+3)}}\right)\\
&\quad=\left(\frac{-\beta\omega p^{\frac{1}{4}(n'-n-1)}}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{1}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;pq^3,\frac{\alpha\beta\omega^2p^{\frac{1}{4}(2n'-1)}}{q^{\frac{1}{4}(2n+3)}}\right),
\end{aligned}\tag{27}$$

$$\begin{aligned}
&T^{(E-,E+)}_{n'n}(\alpha,\beta)\\
&\quad=\left(\frac{-\beta\omega p^{\frac{1}{2}n}}{q^{\frac{1}{4}(n+n'+1)}(p^{1/2}-q^{-1/2})}\right)^{n-n'}\frac{1}{[n-n']_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n-n'+1\end{matrix};p,q;p^3q,\frac{\alpha\beta\omega^2p^{\frac{4n-2n'-1}{4}}}{q^{\frac{2n+3}{4}}}\right)\\
&\quad=\left(\frac{-\alpha\omega p^{\frac{1}{4}(n'-n-1)}}{p^{1/2}-q^{-1/2}}\right)^{n'-n}\frac{1}{[n'-n]_{p,q}!}\,{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ n'-n+1\end{matrix};p,q;p^3q,\frac{\alpha\beta\omega^2p^{\frac{4n'-2n-1}{4}}}{q^{\frac{2n'+3}{4}}}\right).
\end{aligned}$$

The region of convergence for the matrix elements can easily be determined. We now show that each of these families of matrix elements determines two variable models of the irreducible representations $Q(\omega)$.

4. Two-variable models of $\mathcal{E}_{p,q}(2)$ representations

As a consequence of the commutation relations (13), the matrix elements themselves define two-variable models of the representations $Q(\omega)$. For the matrix elements $(e+,e-)$, we note that the operator identities

$$\begin{aligned}
&e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)P_+=\frac{1}{q^{1/2}\beta}\left(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\right)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-),\\
&e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)P_-=\frac{1}{q^{1/2}\alpha}\left(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\right)e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-),\\
&\left[J,e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)\right]=(\beta\partial_\beta-\alpha\partial_\alpha)e_{p,q}(\beta P_+)e_{p,q}(\alpha P_-),
\end{aligned}\tag{28}$$

imply

$$\begin{aligned}
&\omega T^{(e+,e-)}_{n',n+1}(\alpha,\beta)=\frac{1}{q^{1/2}\beta}\left(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\right)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}T^{(e+,e-)}_{n'n}(\alpha,\beta),\\
&\omega\left(\frac{p}{q}\right)^{n/2}T^{(e+,e-)}_{n',n-1}(\alpha,\beta)=\frac{1}{q^{1/2}\alpha}\left(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\right)T^{(e+,e-)}_{n'n}(\alpha,\beta),\\
&(n-n')T^{(e+,e-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(e+,e-)}_{n'n}(\alpha,\beta).
\end{aligned}\tag{29}$$

For the matrix elements $(e-,e+)$, the operator identities

$$\begin{aligned}
&e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)P_+=\frac{1}{q^{1/2}\alpha}\left(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\right)e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+),\\
&e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)P_-=\frac{1}{q^{1/2}\beta}\left(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\right)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+),\\
&\left[J,e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)\right]=(\alpha\partial_\alpha-\beta\partial_\beta)e_{p,q}(\beta P_-)e_{p,q}(\alpha P_+),
\end{aligned}\tag{30}$$

imply

$$\omega T^{(e-,e+)}_{n',n+1}(\alpha,\beta)=\frac{1}{q^{1/2}\alpha}\big(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\big)T^{(e-,e+)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(e-,e+)}_{n',n-1}(\alpha,\beta)=\frac{1}{q^{1/2}\beta}\big(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}T^{(e-,e+)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(e-,e+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(e-,e+)}_{n'n}(\alpha,\beta). \tag{31}$$

For the matrix elements $(e+,E-)$, the operator identities

$$e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)P_+=\frac{1}{q^{1/2}\beta}\big(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-),$$
$$e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)P_-=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-),$$
$$\big[J,e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)\big]=(\beta\partial_\beta-\alpha\partial_\alpha)e_{p,q}(\beta P_+)E_{p,q}(\alpha P_-), \tag{32}$$

yield the relations

$$\omega T^{(e+,E-)}_{n',n+1}(\alpha,\beta)=\frac{1}{q^{1/2}\beta}\big(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}T^{(e+,E-)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(e+,E-)}_{n',n-1}(\alpha,\beta)=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)T^{(e+,E-)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(e+,E-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(e+,E-)}_{n'n}(\alpha,\beta), \tag{33}$$

and for the matrix elements $(e-,E+)$, the operator identities

$$e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)P_+=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+),$$
$$e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)P_-=\frac{1}{q^{1/2}\beta}\big(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+),$$
$$\big[J,e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)\big]=(\alpha\partial_\alpha-\beta\partial_\beta)e_{p,q}(\beta P_-)E_{p,q}(\alpha P_+), \tag{34}$$

imply

$$\omega T^{(e-,E+)}_{n',n+1}(\alpha,\beta)=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)T^{(e-,E+)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(e-,E+)}_{n',n-1}(\alpha,\beta)=\frac{1}{q^{1/2}\beta}\big(I-T^{1/2}_{p,\beta}T^{1/2}_{q,\beta}\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}T^{(e-,E+)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(e-,E+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(e-,E+)}_{n'n}(\alpha,\beta). \tag{35}$$

For the matrix elements $(E+,e-)$, the operator identities

$$E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)P_+=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-),$$
$$E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)P_-=\frac{1}{q^{1/2}\alpha}\big(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\big)E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-),$$
$$\big[J,E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-)\big]=(\beta\partial_\beta-\alpha\partial_\alpha)E_{p,q}(\beta P_+)e_{p,q}(\alpha P_-), \tag{36}$$

imply

$$\omega T^{(E+,e-)}_{n',n+1}(\alpha,\beta)=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}T^{(E+,e-)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(E+,e-)}_{n',n-1}(\alpha,\beta)=\frac{1}{q^{1/2}\alpha}\big(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\big)T^{(E+,e-)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(E+,e-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(E+,e-)}_{n'n}(\alpha,\beta), \tag{37}$$

and for $(E-,e+)$, the operator identities

$$E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)P_+=\frac{1}{q^{1/2}\alpha}\big(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\big)E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+),$$
$$E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)P_-=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+),$$
$$\big[J,E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+)\big]=(\alpha\partial_\alpha-\beta\partial_\beta)E_{p,q}(\beta P_-)e_{p,q}(\alpha P_+), \tag{38}$$

imply

$$\omega T^{(E-,e+)}_{n',n+1}(\alpha,\beta)=\frac{1}{q^{1/2}\alpha}\big(I-T^{1/2}_{p,\alpha}T^{1/2}_{q,\alpha}\big)T^{(E-,e+)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(E-,e+)}_{n',n-1}(\alpha,\beta)=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}T^{(E-,e+)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(E-,e+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(E-,e+)}_{n'n}(\alpha,\beta). \tag{39}$$

Similarly, the operator identities

$$E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)P_+=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-),$$
$$E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)P_-=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-),$$
$$\big[J,E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-)\big]=(\beta\partial_\beta-\alpha\partial_\alpha)E_{p,q}(\beta P_+)E_{p,q}(\alpha P_-) \tag{40}$$

for the matrix elements $(E+,E-)$, imply

$$\omega T^{(E+,E-)}_{n',n+1}(\alpha,\beta)=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{1/2}_{p,\alpha}T^{-1/2}_{q,\alpha}T^{(E+,E-)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(E+,E-)}_{n',n-1}(\alpha,\beta)=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)T^{(E+,E-)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(E+,E-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(E+,E-)}_{n'n}(\alpha,\beta). \tag{41}$$

Finally, for the matrix elements $(E-,E+)$, the operator identities

$$E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)P_+=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+),$$
$$E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)P_-=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+),$$
$$\big[J,E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+)\big]=(\alpha\partial_\alpha-\beta\partial_\beta)E_{p,q}(\beta P_-)E_{p,q}(\alpha P_+), \tag{42}$$

imply

$$\omega T^{(E-,E+)}_{n',n+1}(\alpha,\beta)=\frac{p^{1/2}}{\alpha}\big(T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}-I\big)T^{(E-,E+)}_{n'n}(\alpha,\beta),$$
$$\omega\left(\frac{p}{q}\right)^{n/2}T^{(E-,E+)}_{n',n-1}(\alpha,\beta)=\frac{p^{1/2}}{\beta}\big(T^{-1/2}_{p,\beta}T^{1/2}_{q,\beta}-I\big)T^{-1/2}_{p,\alpha}T^{1/2}_{q,\alpha}T^{(E-,E+)}_{n'n}(\alpha,\beta),$$
$$(n-n')T^{(E-,E+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(E-,E+)}_{n'n}(\alpha,\beta). \tag{43}$$

Thus, using the notations

$$\begin{aligned}
&\widetilde{P}^x = \frac{1}{q^{1/2}x}\big(I - T_{p,x}^{1/2}T_{q,x}^{1/2}\big),\\
&\widehat{P}^x = T_{p,x}^{-1/2}T_{q,x}^{1/2},\\
&\widetilde{J} = \alpha\partial_\alpha - \beta\partial_\beta,
\end{aligned}\tag{44}$$

we see that the following sets of operators and basis vectors define a two-variable realization of relations (14) in each case. Hence a model of the representation $Q(\omega)$ is any one out of the following eight realizations:

$$\begin{aligned}
&\text{(i)} && \widetilde{P}^\beta\big(\widehat{P}^\alpha\big)^{-1}, \widetilde{P}^\alpha, \widetilde{J}, && f_{-n'+n} = T_{n'n}^{(e+,e-)}(\alpha,\beta),\\
&\text{(ii)} && \widetilde{P}^\alpha, \widetilde{P}^\beta\widehat{P}^\alpha, -\widetilde{J}, && f_{-n'+n} = T_{n'n}^{(e-,e+)}(\alpha,\beta),\\
&\text{(iii)} && \widetilde{P}^\beta\big(\widehat{P}^\alpha\big)^{-1}, \frac{p^{1/2}}{\alpha}\big(\widehat{P}^\alpha - I\big), \widetilde{J}, && f_{-n'+n} = T_{n'n}^{(e+,E-)}(\alpha,\beta),\\
&\text{(iv)} && \frac{p^{1/2}}{\alpha}\big(\widehat{P}^\alpha - I\big), \widetilde{P}^\beta\widehat{P}^\alpha, -\widetilde{J}, && f_{-n'+n} = T_{n'n}^{(e-,E+)}(\alpha,\beta),\\
&\text{(v)} && \frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\big(\widehat{P}^\alpha\big)^{-1}, \widetilde{P}^\alpha, \widetilde{J}, && f_{-n'+n} = T_{n'n}^{(E+,e-)}(\alpha,\beta),\\
&\text{(vi)} && \widetilde{P}^\alpha, \frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\widehat{P}^\alpha, -\widetilde{J}, && f_{-n'+n} = T_{n'n}^{(E-,e+)}(\alpha,\beta),\\
&\text{(vii)} && \frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\big(\widehat{P}^\alpha\big)^{-1}, \frac{p^{1/2}}{\alpha}\big(\widehat{P}^\alpha - I\big), \widetilde{J}, && f_{-n'+n} = T_{n'n}^{(E+,E-)}(\alpha,\beta),\\
&\text{(viii)} && \frac{p^{1/2}}{\alpha}\big(\widehat{P}^\alpha - I\big), \frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\widehat{P}^\alpha, -\widetilde{J}, && f_{-n'+n} = T_{n'n}^{(E-,E+)}(\alpha,\beta).
\end{aligned}\tag{45}$$

We now show that the addition theorems for the basis vectors $T_{n'n}(\alpha,\beta)$ can be derived by using the relations (29), (31), (33), (35), (37), (39), (41) and (43). For example, consider the model (45)(vi) with raising and lowering operators defined as

$$\begin{aligned}
&E_+ = \widetilde{P}^\alpha,\\
&E_- = \frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\widehat{P}^\alpha,
\end{aligned}$$

applied to the basis vectors $f_{m_0} = T_{-m_0,0}^{(E-,e+)}(\alpha,\beta)$. Using

$$e_{p,q}\big(\gamma\widetilde{P}^\alpha\big)\alpha^m = \alpha^m\sum_{n=0}^{\infty}\big([-m]_{p,q}\big)_n\left(\frac{-\gamma p^{m/2}}{q^{1/2}\alpha}\right)^n(pq)^{-\frac{1}{2}\binom{n}{2}},\tag{46}$$

$$E_{p,q}\left(\xi\frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\widehat{P}^\alpha\right)\alpha^m\beta^n = \alpha^m\beta^n\sum_{l=0}^{\infty}\frac{p^{\frac{1}{2}\binom{l}{2}}}{[l]_{p,q}!}\left(\frac{\xi(p^{\frac{n}{2}} - q^{\frac{n}{2}})q^{\frac{m}{2}}}{\beta(p^{1/2} - q^{-1/2})p^{\frac{m+n-1}{2}}}\right)^l,\tag{47}$$

the addition theorem

$$e_{p,q}\big(\gamma\widetilde{P}^\alpha\big)E_{p,q}\left(\xi\frac{p^{1/2}}{\beta}\big(\widehat{P}^\beta - I\big)\widehat{P}^\alpha\right)T_{-m_0,0}^{(E-,e+)}(\alpha,\beta) = \sum_{n=-\infty}^{\infty}T_{n0}^{(e+,E-)}(\xi,\gamma)T_{-m_0,n}^{(E-,e+)}(\alpha,\beta),\tag{48}$$

takes the following form:

$$\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{m_0}\frac{q^{\frac{1}{2}\binom{m_0}{2}}}{[m_0]_{p,q}!}\sum_{k=0}^{\infty}\frac{(\alpha\beta\omega^2 p^{m_0/2})^k(\frac{p}{q})^{k^2/2}}{(p^{1/2}-q^{-1/2})^{2k}[k]_{p,q}!([m_0+1]_{p,q})_k}E_{p,q}\left(\frac{-\xi(p^{\frac{k+m_0}{2}}-q^{\frac{k+m_0}{2}})q^{\frac{k}{2}}}{\beta p^{\frac{2k+m_0-1}{2}}}\right)$$
$$\times{}_1\widetilde{\psi}_0\left(\begin{matrix}-k\\ -\end{matrix};p,q;(pq)^{-1},\frac{-\gamma p^{\frac{2k+1}{4}}}{\alpha}\right)$$
$$=\sum_{n=-\infty}^{\infty}T_{n0}^{(e+,E-)}(\xi,\gamma)T_{-m_0,n}^{(E-,e+)}(\alpha,\beta). \tag{49}$$

Similarly, as another example, if we consider the model (45)(ii) with basis vectors $f_{m_0}=T_{-m_0,0}^{(e-,e+)}(\alpha,\beta)$, then we get the following addition theorem:

$$E_{p,q}(\gamma\widetilde{P}^{\alpha})E_{p,q}(\xi\widetilde{P}^{\beta}\widehat{P}^{\alpha})T_{-m_0,0}^{(e-,e+)}(\alpha,\beta)=\sum_{n=-\infty}^{\infty}T_{n0}^{(E+,E-)}(\xi,\gamma)T_{-m_0,n}^{(e-,e+)}(\alpha,\beta), \tag{50}$$

which takes the form:

$$\left(\frac{-\beta\omega}{p^{1/2}-q^{-1/2}}\right)^{m_0}\frac{p^{-\frac{1}{2}\binom{m_0}{2}}}{[m_0]_{p,q}!}\sum_{k=0}^{\infty}\frac{(\alpha\beta\omega^2 p^{1/4})^k(\frac{p}{q^3})^{\frac{k^2}{4}}}{(p^{1/2}-q^{-1/2})^{2k}[k]_{p,q}!([m_0+1]_{p,q})_k}{}_1\psi_0\left(\begin{matrix}-k-m_0\\ -\end{matrix};p,q;\frac{-\xi p^{\frac{m_0}{2}}q^{\frac{k-1}{2}}}{\beta}\right)$$
$$\times{}_1\psi_0\left(\begin{matrix}-k\\ -\end{matrix};p,q;\frac{-\gamma p^{\frac{k}{2}}}{q^{1/2}\alpha}\right)$$
$$=\sum_{n=-\infty}^{\infty}T_{n0}^{(E+,E-)}(\xi,\gamma)T_{-m_0,n}^{(e-,e+)}(\alpha,\beta). \tag{51}$$

5. Biorthogonality relations

With the help of the identities (17)–(20) for the matrix elements, we can derive the biorthogonality relations. From (17), we get the following biorthogonality relation,

$$\sum_{l=-\infty}^{\infty}\frac{(zq^{\frac{1}{4}(2n'-2n+1)}p^{-\frac{1}{4}(2n+1)})^l}{(p^{1/2}-q^{-1/2})^{2l}[l-n']_{p,q}![l-n]_{p,q}!}{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n'+1\end{matrix};p,q;\frac{q}{p},z\right)$$
$$\times{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n+1\end{matrix};p,q;p^3q,zq^{\frac{1}{2}(l-n)}p^{l-\frac{1}{2}(n+n'+1)}\right)$$
$$=\frac{(-z)^n}{(p^{1/2}-q^{-1/2})^{2n}}\delta_{n'n}. \tag{52}$$

Similarly, from (18), (19) and (20) we obtain the following biorthogonality relations

$$\sum_{l=-\infty}^{\infty}\frac{(-z)^l}{(p^{1/2}-q^{-1/2})^{2l}}\frac{p^{\frac{l}{4}(2n'-2n-1)}q^{\frac{l^2}{4}-l(n'-\frac{1}{4})}}{[l-n']_{p,q}![l-n]_{p,q}!}$$
$$\times{}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n'+1\end{matrix};p,q;\frac{p}{q},z\right){}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n+1\end{matrix};p,q;pq^3,\frac{zq^l}{p^{1/2}q^{\frac{1}{2}(n+n'+1)}}\right)$$
$$=\frac{(-z)^n}{(p^{1/2}-q^{-1/2})^{2n}}\left(\frac{q}{p}\right)^{n/4}q^{-n^2/2}\delta_{n'n}, \tag{53}$$

$$\sum_{l=-\infty}^{\infty}\left(\frac{-z}{p^{\frac{1}{4}(2n+1)}q^{\frac{1}{4}(2n'-1)}(p^{1/2}-q^{-1/2})^2}\right)^l\frac{(pq)^{l^2/4}}{[l-n']_{p,q}![l-n]_{p,q}!}$$
$$\times\ {}_0\psi_1\left(\begin{matrix}-\\ l-n'+1\end{matrix};p,q;zp^{\frac{1}{2}(l-n')}\right){}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n+1\end{matrix};p,q;q^2,zq^{\frac{1}{2}(l-n)}\right)$$
$$=\frac{(-z)^n}{(p^{1/2}-q^{-1/2})^{2n}p^{n(n+1)/4}q^{n(n-1)/4}}\,\delta_{n'n},\tag{54}$$

$$\sum_{l=-\infty}^{\infty}\frac{(-z)^l q^{l/4}}{(p^{1/2}-q^{-1/2})^{2l}}\frac{(pq)^{l^2/4}}{p^{\frac{l}{4}(2n'+2n+1)}[l-n']_{p,q}![l-n]_{p,q}!}$$
$$\times\ {}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n'+1\end{matrix};p,q;q^2,zq^{\frac{1}{2}(l-n')}\right){}_0\widetilde{\psi}_1\left(\begin{matrix}-\\ l-n+1\end{matrix};p,q;p^2,zp^{\frac{1}{2}(l-n)}\right)$$
$$=\frac{(-z)^n}{(p^{1/2}-q^{-1/2})^{2n}}\left(\frac{q}{p}\right)^{n/4}(pq)^{-n^2/4}\,\delta_{n'n},\tag{55}$$

respectively.

References

[1] I.M. Burban, A.U. Klimyk, p,q-Differentiation, p,q-integration and p,q-hypergeometric functions related to quantum groups, Integral Transforms Spec. Funct. 2 (1994) 15–36.

[2] R. Chakrabarti, R. Jagannathan, A (p,q)-oscillator realization of two parameter quantum algebras, J. Phys. A 24 (1991) L711–L718.

[3] R. Floreanini, L. Vinet, Representations of quantum algebras and q-special functions, in: Quantum Symmetries, Clausthal, 1991, World Sci. Publishing, River Edge, NJ, 1993, pp. 264–284.

[4] R. Floreanini, L. Lapointe, L. Vinet, A note on (p,q)-oscillators and bibasic hypergeometric functions, J. Phys. A 26 (1993) L611–L614.

[5] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge Univ. Press, Cambridge, UK, 2004.

[6] R. Jagannathan, K. Srinivasa Rao, Two-parameter quantum algebra, twin-basic numbers, and associated generalized hypergeometric series, in: Proceedings of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India, 20–21 December 2005.

[7] E.G. Kalnins, W. Miller, S. Mukherjee, Models of q-algebra representations: Matrix elements of q-oscillator algebra, J. Math. Phys. 43 (1993) 5333–5356.

[8] E.G. Kalnins, W. Miller, S. Mukherjee, Models of q-algebra representations: The group of plane motions, SIAM J. Math. Anal. 25 (1994) 513–527.

[9] E.G. Kalnins, W. Miller, S. Mukherjee, Models of q-algebra representations: Matrix elements of $U_q(su_2)$, in: Contemp. Math., vol. 160, Amer. Math. Soc., Providence, RI, 1994, pp. 191–208.

[10] W. Miller, Lie Theory and Special Functions, Academic Press, New York, 1968.

[11] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p,q-special functions, J. Math. Anal. Appl. 335 (2007) 268–279.